## CHAPTER 8 Inverse and Square Matrices

Several topics in this chapter are not essential so selection may be necessary. However inverse matrices are needed for later work on transformations.

### EXERCISE 8a (p. 132)

**1.** Yes, 3 x 3
**2.** No
**3.** Yes, 2 x 2
**4.** No
**5.** Yes, 2 x 2
**6.** Yes, 3 x 3

### EXERCISE 8b (p. 133)

**1.** $\begin{pmatrix} 4 & 7 \\ 7 & 11 \end{pmatrix}$

**2.** $\begin{pmatrix} 7 & -6 \end{pmatrix}$

**3.** $\begin{pmatrix} 7 & 10 & 1 \\ 15 & 26 & 1 \end{pmatrix}$

**4.** Not possible

**5.** Not possible

**6.** $\begin{pmatrix} 26 & 13 \\ -4 & -2 \end{pmatrix}$

**7.** $\begin{pmatrix} 11 & 7 \\ 7 & 4 \end{pmatrix}$

**8.** Not possible

**9.** $\begin{pmatrix} 4 & 24 & 4 \\ 3 & 18 & 3 \\ 2 & 12 & 2 \end{pmatrix}$

**10.** (24)

**11.** $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

**12.** (34 6)

### EXERCISE 8c (p. 134)

**1.** $\begin{pmatrix} 4 & 2 \\ 3 & 4 \end{pmatrix}$

**2.** $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$

**3.** $\begin{pmatrix} 0 & 0 \end{pmatrix}$

**4.** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

**5.** $\begin{pmatrix} 3 & 2 \end{pmatrix}$

**6.** $\begin{pmatrix} 3 & 2 & -1 \\ 4 & 3 & 1 \end{pmatrix}$

### EXERCISE 8d (p. 135)

**1.** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

**2.** $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

**3.** $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

**4.** $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

**5.** $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

**6.** $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

**7.** $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

**8.** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

**10.** $\begin{pmatrix} 6 & -2 \\ -8 & 3 \end{pmatrix}$

**11.** $\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}$

**12.** $\begin{pmatrix} 3 & 1 \\ 20 & 6 \end{pmatrix}$

**13.** $\begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix}$

**14.** $\begin{pmatrix} 4 & -7 \\ -3 & 5 \end{pmatrix}$ **16.** $\begin{pmatrix} 8 & -4 \\ -9 & 5 \end{pmatrix}$ **18.** $\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$

**15.** $\begin{pmatrix} 6 & -3 \\ -9 & 5 \end{pmatrix}$ **17.** $\begin{pmatrix} 4 & -2 \\ -14 & 6 \end{pmatrix}$ **19.** $\begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix}$

**EXERCISE 8e (p. 137)**

**1.** $\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ **3.** $\begin{pmatrix} -2 & 3 \\ -7 & 10 \end{pmatrix}$ **5.** $\begin{pmatrix} 7 & -4 \\ -12 & 7 \end{pmatrix}$

**2.** $\begin{pmatrix} 2 & -3 \\ -7 & 11 \end{pmatrix}$ **4.** $\begin{pmatrix} 7 & -5 \\ -4 & 3 \end{pmatrix}$ **6.** $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

**EXERCISE 8f (p. 138)**

Before Number 16, ask the pupils to try to find the inverse of, say, $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$ and discuss again the fact that division by zero is impossible—hence no inverse.

**1.** $\begin{pmatrix} 1\frac{1}{2} & -1 \\ -4 & 3 \end{pmatrix}$

**2.** $\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} \\ -1 & 3 \end{pmatrix}$

**3.** $\begin{pmatrix} 1\frac{1}{2} & -\frac{1}{2} \\ -2\frac{1}{2} & 1 \end{pmatrix}$

**4.** $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$

**5.** $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$

**6.** $\begin{pmatrix} 4 & -1 \\ -5\frac{1}{2} & -1\frac{1}{2} \end{pmatrix}$

**7.** $\begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}$

**8.** $\begin{pmatrix} 4 & -3 \\ -5 & 4 \end{pmatrix}$

**9.** $\begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{3}{4} \end{pmatrix}$

**10.** $\begin{pmatrix} -1 & 1 \\ 3\frac{1}{2} & -3 \end{pmatrix}$

**11.** $\begin{pmatrix} -1 & 2 \\ 2\frac{1}{2} & -4\frac{1}{2} \end{pmatrix}$

**12.** $\begin{pmatrix} -\frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$

**13.** $\begin{pmatrix} -1 & 1 \\ 2 & -1\frac{1}{2} \end{pmatrix}$

**14.** $\begin{pmatrix} -1 & -1\frac{1}{3} \\ -1 & -1 \end{pmatrix}$

**15.** $\begin{pmatrix} -3 & 2 \\ 4 & -2\frac{1}{2} \end{pmatrix}$

**16.** a) Yes b) No c) Yes

**17.** a) Yes b) Yes c) Yes

**18.** $\begin{pmatrix} 1 & -1 \\ -1 & 1\frac{1}{5} \end{pmatrix}$

**19.** No inverse

**20.** $\begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$

**21.** $\begin{pmatrix} -4 & 7 \\ 3 & -5 \end{pmatrix}$

**22.** $\begin{pmatrix} 2 & -1 \\ -3 & 1\frac{2}{3} \end{pmatrix}$

**23.** No inverse

**EXERCISE 8g (p. 140)**

**1.** 1, $\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$

**2.** 2, $\begin{pmatrix} 1\frac{1}{2} & -1 \\ -1 & 1 \end{pmatrix}$

**3.** **I**

**4.** $\begin{pmatrix} 16 & 19 \\ 10 & 12 \end{pmatrix}$

**5.** $\begin{pmatrix} 6 & -9\frac{1}{2} \\ -5 & 8 \end{pmatrix}$

**6.** $\begin{pmatrix} 6 & -5 \\ -9\frac{1}{2} & 8 \end{pmatrix}$

**7.** $\begin{pmatrix} 6 & -9\frac{1}{2} \\ -5 & 8 \end{pmatrix}$

**8.** $\begin{pmatrix} 34 & 21 \\ 21 & 13 \end{pmatrix}$

**9.** $\begin{pmatrix} 13 & -21 \\ -21 & 34 \end{pmatrix}$

**10.** $\begin{pmatrix} 13 & -21 \\ -21 & 34 \end{pmatrix}$

**11.** $\begin{pmatrix} 2 & 1 \\ -3\frac{1}{2} & -1\frac{1}{2} \end{pmatrix}$

**12.** $\begin{pmatrix} -2\frac{1}{2} & 4 \\ -2 & 3 \end{pmatrix}$

**EXERCISE 8h (p. 142)**

The formula for finding the value of |**A**| is not essential and none of the questions in the rest of this chapter depends upon it.

**1.** 9
**2.** 17
**3.** 0
**4.** 19
**5.** –14
**6.** 10
**7.** –1
**8.** –8
**9.** 9
**10.** 5
**11.** 5
**12.** –9

**EXERCISE 8i (p. 143)**

Solution of simultaneous equations by elimination demands that decisions are made at several stages. Pupils may notice that using matrices to solve simultaneous equations is not as neat as the elimination method and generally takes longer. This is a good time to explain that, because no decisions have to be made when using matrices, it is an ideal method for computer programming.

**1.** $x + 2y = 3$
$3x + 2y = 5$

**2.** $4x + 2y = 12$
$5x + 3y = 15$

**3.** $9x + 2y = 24$
$4x + y = 11$

**4.** $6p - q = -8$
$2p + q = 0$

**5.** $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$

**6.** $\begin{pmatrix} 4 & -3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

**7.** $\begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

**8.** $\begin{pmatrix} 3 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

**9.** $\begin{pmatrix} 7 & -2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \end{pmatrix}$

**10.** $\begin{pmatrix} 5 & 1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -8 \\ -14 \end{pmatrix}$

**EXERCISE 8j (p.145)**

**1.** $x = 1,\ y = 2$
**2.** $x = 2,\ y = 3$
**3.** $x = 1,\ y = -1$
**4.** $x = 2,\ y = -1$
**5.** $x = 3,\ y = 0$
**6.** $x = 1,\ y = 2$
**7.** $x = 4,\ y = 2$
**8.** $x = 1,\ y = -2$
**9.** $x = 4,\ y = 2$
**10.** $p = 1,\ q = 1$
**11.** $s = -2,\ t = 3$

**12.** $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$; $x = 1, y = 1$

**13.** $\begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$; $x = 2, y = 3$

**14.** $\begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $x = 1, y = -1$

**15.** $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 23 \end{pmatrix}$; $x = 6, y = 1$

**16.** $\begin{pmatrix} 9 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \end{pmatrix}$; $x = \frac{1}{3}, y = 4$

**17.** $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$; $x = 2, y = 1$

**18.** $\begin{pmatrix} 5 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 16 \\ 3 \end{pmatrix}$; $x = 2, y = 3$

**19.** $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \end{pmatrix}$; $x = -1, y = 3$

**20.** Determinant is zero so there is no inverse.
**21.** Determinant is zero so there is no inverse.

**EXERCISE 8k (p. 147)**

**1.** $\begin{pmatrix} 5 & 6 \\ -3 & 0 \end{pmatrix}$

**2.** $\begin{pmatrix} 7 & 2 \\ -3 & 2 \end{pmatrix}$

**3.** $\begin{pmatrix} 15 & 19 \\ 9 & 9 \end{pmatrix}$

**4.** $\begin{pmatrix} -5 & -6 \\ 3 & 0 \end{pmatrix}$

**5.** $\begin{pmatrix} 2 & 1\frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}$

**6.** $\begin{pmatrix} 3 & -6 \\ 0 & 3 \end{pmatrix}$

**7.** $\begin{pmatrix} 3 & -1 \\ -5 & 1 \end{pmatrix}$

**8.** $\begin{pmatrix} -\frac{1}{2} & 1\frac{1}{2} \\ 1 & -2 \end{pmatrix}$

**EXERCISE 8l (p. 147)**

**1.** $\begin{pmatrix} 5 & 4 & 3 \\ 10 & -8 & 4 \end{pmatrix}$

**2.** $\begin{pmatrix} 1 & 3\frac{1}{2} \\ 1\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

**3.** 24

**4.** $\begin{pmatrix} \frac{4}{7} & -\frac{3}{7} \\ -\frac{3}{7} & \frac{4}{7} \end{pmatrix}$

**5.** $\begin{pmatrix} -9 \end{pmatrix}$

**6.** $\begin{pmatrix} 13 & 33 \\ 6 & 22 \end{pmatrix}$

**EXERCISE 8m (p. 148)**

**1.** $\begin{pmatrix} 5 & 3 \\ -1 & 4 \end{pmatrix}$

**2.** 2

**3.** $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

**4.** $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

**5.** $\begin{pmatrix} 6 & 10 \end{pmatrix}$

**6.** $\begin{pmatrix} 3 & -1\frac{1}{2} \end{pmatrix}$

**Codes:** The following is a fun way of using matrices and gives extra practice in the use of inverses. It does take a long time though, especially with those pupils who are careless!

We can use a 2 x 2 matrix to code a message and we can use its inverse for decoding. Choose a matrix with a determinant of 1 so that the entries in the inverse are whole numbers: for example $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ has as its inverse $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

Give to each letter of the message a number according to its position in the alphabet.

| G | O | | A | W | A | Y |
|---|---|---|---|---|---|---|
| 7 | 15 | | 1 | 23 | 1 | 25 |

Make the number of letters up to a multiple of 4 by adding "A"s.

| G | O | | A | W | A | Y | A | A |
|---|---|---|---|---|---|---|---|---|
| 7 | 15 | | 1 | 23 | 1 | 25 | 1 | 1 |

Now we can form two 2 x 2 matrices from these numbers, i.e. $\begin{pmatrix} 7 & 15 \\ 1 & 23 \end{pmatrix}$ and $\begin{pmatrix} 1 & 25 \\ 1 & 1 \end{pmatrix}$.

Premultiply each by the coding matrix.

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 7 & 15 \\ 1 & 23 \end{pmatrix} = \begin{pmatrix} 15 & 53 \\ 8 & 38 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 25 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 51 \\ 2 & 26 \end{pmatrix}$$

The coded message is 15, 53, 8, 38, 3, 51, 2, 26.

To decode the message we form matrices again from the coded message and use the decoder, that is, the inverse matrix. This gives the original numbers.

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 15 & 53 \\ 8 & 38 \end{pmatrix} = \begin{pmatrix} 7 & 15 \\ 1 & 23 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 51 \\ 2 & 26 \end{pmatrix} = \begin{pmatrix} 1 & 25 \\ 1 & 1 \end{pmatrix}$$

The following messages have been coded using the given matrices.

1. GOODBYE $\begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}$ –1, 26, 6, 41, –1, 49, 1, 74

| | | |
|---|---|---|
| 2. HAPPY BIRTHDAY | $\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ | 88, 67, 64, 50, 111, 78, 77, 58, 76, 28, 52, 19, 79, 7, 53, 5 |
| 3. JACK AND JILL | $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ | 45, 58, 16, 23, 23, 92, 9, 34, 87, 41, 33, 14 |
| 4. GEOMETRY | $\begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}$ | 133, 111, 37, 31, 146, 255, 41, 70 |
| 5. HULLO | $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ | 28, 54, 20, 33, 31, 3, 16, 2 |